Last time:

* Motional emf: Induced on a conductor that moves through a magnetic field

$$\mathcal{E} = \oint \underline{F}_B \cdot d\underline{s}$$

$$\boxed{\mathcal{E} = -\frac{\partial \Phi_B}{\partial t}}$$

where $\Phi_B = \int \underline{B} \cdot d\underline{a}$ magnetic flux through loop

* Lenz law: Direction of induced current

induced current opposes a change in flux of $\underline{B}$

* Faraday's law: A changing magnetic field induces an electric field.

$$\oint \underline{E} \cdot d\underline{l} = -\int \frac{\partial \underline{B}}{\partial t} \cdot d\underline{a} = -\frac{d\Phi}{dt}$$

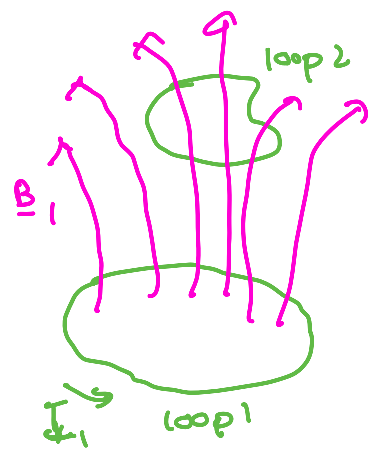
$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

The end of electrostatics

$\oint \underline{E} \cdot d\underline{l} \neq 0$

$\nabla \times \underline{E} \neq 0$

Today: Inductance and magnetic energy



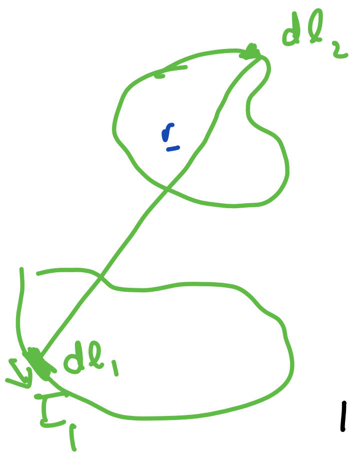
A change in Φ_B will give rise to an emf. Assume we have 2 loops close to each other. What is the emf induced in one loop due to a change in current in the other?

* Run I_1 through loop 1, it produces a magnetic field $\underline{B}_1$.

* Field lines from $\underline{B}_1$ will pass through loop 2 $\rightarrow$ change in flux of $\underline{B}$ $\rightarrow$ induced emf $\rightarrow$ induced current.

How much current will be induced?

Let's look again at the loops:



To know the current induced we would need to know $\underline{B}_1$ due to I_1

From Biot-Savart's law:

$$\underline{B}_1 = \frac{\mu_0}{4\pi} I_1 \int \frac{d\underline{\ell}_1 \times \hat{r}}{r^2}$$

$\underline{B}_1$ is proportional to I_1

If $\underline{B}_1$ is proportional to $I_1 \Rightarrow$ the flux through loop 2 will be proportional to I_1.

$$\Phi_2 = \int \underline{B}_1 \cdot d\underline{a}_2 = \underbrace{M_{21}}_{\equiv \text{ Mutual inductance: constant of proportionality}} I_1$$

Φ_2 is proportional to I_1

We can derive a formula for M_{21} by expressing Φ_2 in terms of the vector potential & using Stokes theorem:

$$\Phi_2 = \int \underline{B}_1 \cdot d\underline{a}_2 = \int (\nabla \times A_1) \cdot d\underline{a}_2 \underset{\text{Stokes theorem}}{=} \oint \underline{A}_1 \cdot d\underline{\ell}_2 \quad *$$

From lecture 15, for a loop of current A_1:

$$\underline{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\underline{\ell}_1}{r} \quad \text{, subs in } *:$$

$$\Phi_2 = \frac{\mu_0 I_1}{4\pi} \oint \left(\frac{d\underline{\ell}_1}{r}\right) \cdot d\underline{\ell}_2 = M_{21} I_1$$

$$\therefore \boxed{M_{21} = \frac{\mu_0}{4\pi} \oint\oint \frac{d\underline{\ell}_1 \cdot d\underline{\ell}_2}{r}}$$

Neumann formula

* M_{21} is purely geometrical, depends on dl_1 & dl_2 (similar to capacitance)

* If we switch the roles of loop 1 and loop 2, it remains unchanged

$$M_{21} = M_{12} = M \equiv \text{mutual inductance.}$$

As we vary the current on loop 1, we get a change in flux in loop 2. that results in an emf.

$$\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -M\frac{dI_1}{dt} \qquad \text{since } \Phi_2 = M I_1$$

This will induce a current on loop 2 given by Ohm's law

$$\mathcal{E}_2 = R_2 I_2$$

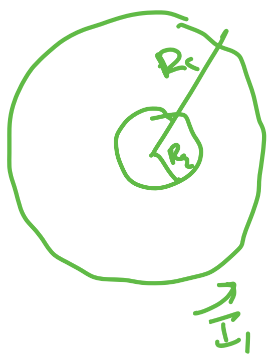
The units for inductance are:

$[M]$ = henry

$$1 \text{ henry} = 1\,H = \frac{T \cdot m^2}{A} = \frac{V \cdot s}{A}$$

where T = tesla
m = meter
A = ampère.

Example: What is the mutual inductance of 2 concentric, coplanar loops of radius R_1 and R_2. with $R_1 \gg R_2$



Mutual inductance is the proportionality constant between Φ_2 and I_1

$$\Phi_2 = M I_1$$

we need Φ_2, we need B_1

The converse is also true. $\Phi_1 = M \Phi_2$ we would need B_2 to calculate Φ_1, in this case this is more complicated than B_1.

We will calculate B_1. In lecture 14 we calculated the magnetic field of a loop of current at a point in the z axis.

$$B_z(z) = \frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}} \qquad \text{at } z=0$$

$$B_0 = \frac{\mu_0 I}{2R}$$

The magnetic field due to I_1 at the center of the outer loop is:

$$B_1 = \frac{\mu_0 I_1}{2R_1}$$ we can do this since $R_1 >> R_2$

So the flux through the loop is

$$\Phi_2 = B_1 A_2 = \left(\frac{\mu_0 I}{2R_1}\right)\pi R_2^2 = \frac{\mu_0 \pi I_1 R_2^2}{2R_1}$$

$$\Phi = B_1 \int da_2$$

By definition $\Phi_2 = M I_1 \Rightarrow M = \frac{\Phi_2}{I_1}$

$$\therefore M = \frac{\mu_0 \pi \cancel{I_1} R_2^2}{2R_1 \cancel{I_1}} = \boxed{\frac{\mu_0 \pi R_2^2}{2R_1}}$$

independent of current, depends only on geometrical quantities.

Self inductance

A changing current will induce an emf on the source loop itself.

The field and the flux is proportional to the current induced:

$$\Phi = LI$$ where $L \equiv$ self inductance

If the current changes in time, we will have an induced emf on the loop:

$$\mathcal{E} = -L \frac{dI}{dt}$$

Inductance is intrinsically positive.
Emf is in a direction to oppose changes in current.
For this reason we call it back emf.

L measures the "resistance" to a change in current.
the greater L, the harder to change the current.

Example: Calculate the self inductance of a solenoid with N turns, length ℓ and radius R with a current I through each turn.

solved in the pdf

Energy stored in the magnetic field

It takes energy to start current in a circuit with an inductor, an inductor opposes any change in current through it.

$\Leftrightarrow$ we must do work to overcome the back emf.

This is a fixed amount of work and it's recoverable.

We can think of the energy as latent, there is energy stored in the magnetic field. The role of an inductor in the magnetic case is analogous to the capacitor in the electric case.

The work done per unit charge against the back emf, in one trip around the circuit is $-\mathcal{E}$. It has a minus sign since its work done by the outside (you), not the emf.

The amount of charge per unit time passing down the wire is I

$$\frac{dW}{dt} = -\mathcal{E} I = -\left(-L\frac{dI}{dt}\right) I = L I \frac{dI}{dt} *$$

If $\frac{dI}{dt} > 0$ external source is doing positive work to transfer energy to the inductor

current increasing

$\frac{dI}{dt} < 0$ external source takes energy away from the inductor.

current decreasing

We can obtain the total work done by the external source to increase current from zero to I by integrating over time:

$$\int dW = \int_0^I L I' \, dI' \Rightarrow W = \frac{1}{2} L I^2$$

this is equal to the magnetic energy stored in an inductor:

$$\boxed{U_B = \frac{1}{2} L I^2} \; * \quad \left(\text{Recall } U_E = \frac{1}{2}\frac{Q^2}{C} \right)$$

Note: Energy is stored in an inductor NOT dissipated.

This is different from a resistor where energy is dissipated in the form of heat.

Example: Energy stored in a solenoid

A solenoid with length l and radius R consists of N turns of wire. A current passes through it. Find the energy in the system.

$$U_B = \frac{1}{2} L I^2 \; *$$

For a solenoid $L = \mu_0 n^2 l \pi R^2$ here $n = \frac{N}{l}$

Using *:

$$U_B = \frac{1}{2}\left(\mu_0 n^2 l \pi R^2\right) I^2 = \frac{1}{2} \mu_0 n^2 I^2 \pi R^2 l$$

For a solenoid $B = \mu_0 n I$ so in terms of B:

$$U_B = \frac{1}{2\mu_0} (\mu_0 n I)^2 (\pi R^2 l) = \frac{B^2}{2\mu_0} \underbrace{(\pi R^2 l)}_{\text{Volume}}.$$

We can write an expression for the magnetic energy density:

$$u_B = \frac{U_B}{V} = \frac{B^2}{2\mu_0} \quad \text{energy per unit volume}$$

In general

$$\boxed{u_B = \frac{B^2}{2\mu_0}} \quad \text{magnetic energy density} \quad \left(\text{Recall } u_E = \frac{1}{2}\varepsilon_0 E^2 \text{ for electric field.} \right)$$

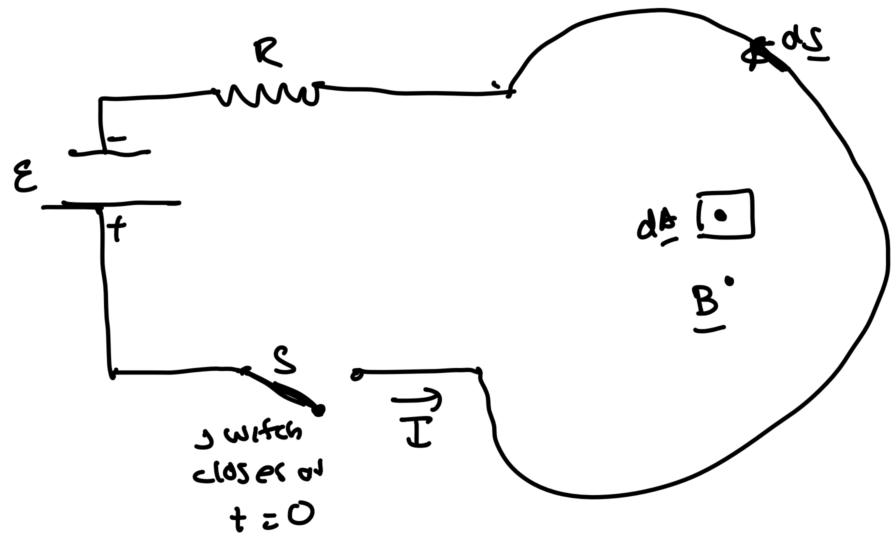
RL circuits

$\oint \underline{E} \cdot d\underline{s} = 0$ For circuits w/ no inductor but this is no longer true, in this case $\oint \underline{E} \cdot d\underline{s} \neq 0$ if we have an inductor:

$$\oint \underline{E} \cdot d\underline{s} = -\frac{d}{dt} \int \underline{B} \cdot d\underline{a} \quad \text{Faraday's law}$$

If we have non-steady currents → we have changing magnetic fields → induced current.

Since $\oint \underline{E} \cdot d\underline{s} \neq 0 \Leftrightarrow \nabla \times \underline{E} \neq 0$ we can no longer define a potential. The electric potential between 2 points in a circuit is no longer well defined.



We have a battery $\mathcal{E}$
a resistor R
a switch S
a one loop inductor
$\underline{B}$ pointing out of the page.

What are the consequences of having the inductor?

At $t=0$ we close the switch

For $t>0$ I will start flowing as shown (from + to − as usual).

What is $I(t)$ for $t>0$?

We need to know what is $\int \underline{E} \cdot d\underline{s}$ around the circuit.

1) The battery $-\mathcal{E}$, $\underline{E}$ is directed from + to −, opposite to the direction of $d\underline{s}$

2) The resistor : There is $\underline{E}$ across the resistor in the direction of the

current $\underline{E}\cdot d\underline{s} > 0$. given by Ohm's law
$\mathcal{E} = IR$ so $+IR$

3) Loop inductor. Since it has no resistance its contribution $\underline{E}\cdot d\underline{s} = 0$

Summing all the contributions

$$\oint \underline{E}\cdot d\underline{s} = -\mathcal{E} + IR$$

What is the flux of magnetic field Φ_B through the surface of the circuit?

We'll consider the area of the resistor + battery negligible compared to the loop inductor

$\Phi_B > 0$ since $\underline{B}$ points out of the page. In the same direction as $d\underline{a}$ $\Rightarrow \underline{B}\cdot d\underline{a} > 0$

From our definition of self inductance

$$\Phi = LI \Rightarrow \frac{d\Phi}{dt} = L\frac{dI}{dt} \quad *$$

Now we'll apply Faraday's law:

$$\oint \underline{E}\cdot d\underline{l} = -\frac{d\Phi}{dt} \quad **$$

Combining * and **

$$\oint \underline{E}\cdot d\underline{l} = -L\frac{dI}{dt} = -\underset{\text{battery}}{\mathcal{E}} + \underset{\text{resistance}}{IR}$$

We can write this equation as:

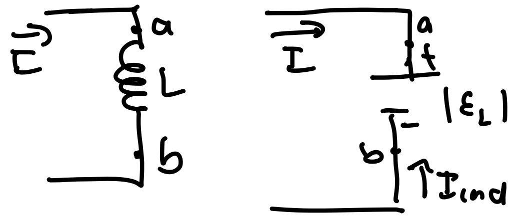
$$\Delta V = \mathcal{E} - IR - L\frac{dI}{dt} = 0$$

Resembles Kirchoff's loop rule
sum of potential drops around a closed loop is zero.

We specify the potential drop across an inductor:

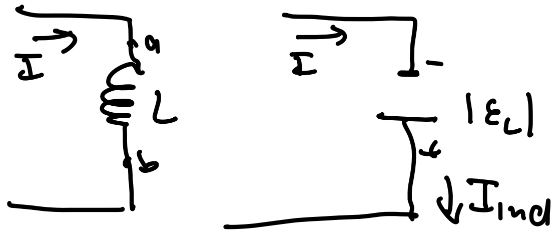
For $\frac{dI}{dt} > 0$ increasing current

For $\frac{dI}{dt} < 0$ decreasing current



$\varepsilon_L = V_b - V_a = -L\left(\frac{dI}{dt}\right) < 0$

I_{ind} will be opposite in direction to I



$\varepsilon_L = V_b - V_a = -L\left(\frac{dI}{dt}\right) > 0$

I_{ind} will be in the same direction as I

In both cases the change in potential along the direction of I is

$$V_b - V_a = -L\left(\frac{dI}{dt}\right)$$

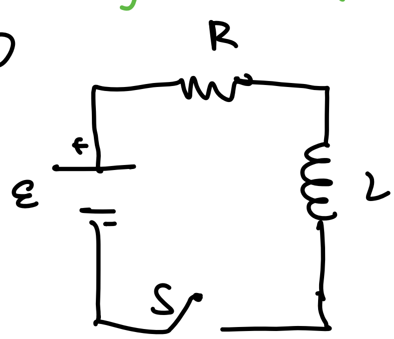
Kirchhoff's loop rule modified for inductors

If an inductor is traversed in the direction of the current, the potential change is $-L\left(\frac{dI}{dt}\right)$. If the inductor is traversed opposite to the current, the potential change is $+L\left(\frac{dI}{dt}\right)$

Kirchhoff's loop rule was based on $\oint \vec{E}\cdot d\vec{l} = 0$ which is not true if we have inductors, in fact it is $L\left(\frac{dI}{dt}\right)$.

Consider a rising current RL circuit

$\frac{dI}{dt} > 0$



Close the switch at $t=0$

Current doesn't rise immediately to its max value since we have an inductor.

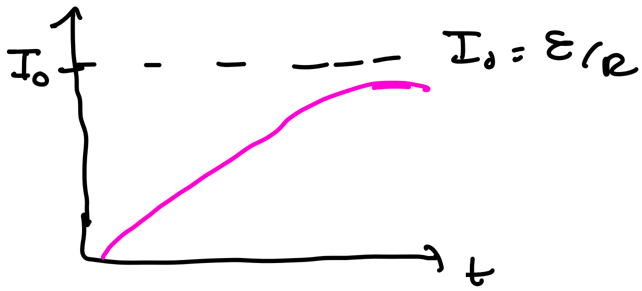
We will solve this circuit using modified Kirchhoff's rule, considering $\frac{dI}{dt} > 0$

$$\varepsilon - IR - L\frac{dI}{dt} = 0$$

$$\Rightarrow \frac{dI}{I - \varepsilon/R} = -\frac{dt}{L/R}$$ integrating both sides, and using that $t = 0$, $I = 0$

$$I(t) = \frac{\varepsilon}{R}\left(1 - e^{-t/\tau}\right)$$ where $\tau = \frac{L}{R}$ time constant of the RL circuit

measures how fast we ~~get~~ the maximum I.



We can calculate the induced emf.

$$|\varepsilon_L| = \left|-L\frac{dI}{dt}\right| = \varepsilon e^{-t/\tau}$$